A discussion of higher-order approximations for the flow field about a slender elliptic cone

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SUMMARY

A comparison is made between various methods for the calculation of the pressure distribution on an elliptic cone in supersonic flow. In particular, a criterion first put forward by Van Dyke (1956) is reconsidered. The paper is presented in two parts under separate authorship, taking the form of a discussion of some controversial aspects of this subject.

PART I. (By R. V.-L.)

In a recent paper Van Dyke (1956) derived the second-order slender-body solution for an unyawed elliptic cone in supersonic flow. Considering his results to be exact, he used them as a criterion for comparing various approximations in compressible flow theory. Among these the method of linearized characteristics (Ferri 1951) was discussed, in particular, its application to the analysis of the flow field about non-circular cones. This method in general proceeds by linearizing the departure between the flow field of interest and some known non-uniform flow close to the actual one; along these lines Ferri treated the particular problem mentioned above by superposing perturbations on the axisymmetric solution, the disturbance velocity components being expanded in Fourier series in the polar angle θ up to the tenth term. With this approach the entire flow field and the shock wave can be determined over a wide range of Mach numbers, in contrast to orthodox methods of linearization.

Van Dyke based his appraisal of Ferri's approximation on the two following numerical comparisons instituted for an elliptic cone of 3 : 1 axis ratio and area equivalent to a 10° circular cone, at $M = \sqrt{2}$:

(1) The law expressing the slender-body pressure distribution was expanded in a Fourier series, and the error introduced by termination of this series at $\cos 10\theta$ was evaluated.

(2) An approximation to the second-order slender-body solution was developed following Ferri's procedure of expanding the velocity components in a Fourier series in θ, while linearizing with respect to deviation of the cross-section from circular, and calculating pressure coefficients from the full relation; the resulting pressures were compared with the exact values.

Van Dyke suggested that the large discrepancies between his complete and approximate results gave a measure of the errors to be encountered in Ferri's method, supposedly due to termination of the series at $\cos 10\theta$ and to linearization of the boundary conditions. Since good accuracy is to be expected from the second-order slender-body theory for the example in question, and since Ferri (1951) reported good agreement with experiments for a similar case, it seemed interesting to ascertain the origin of these discrepancies. The matter has been the subject of private communications with Dr Van Dyke, who has provided many interesting comments and much information; these are gratefully acknowledged.

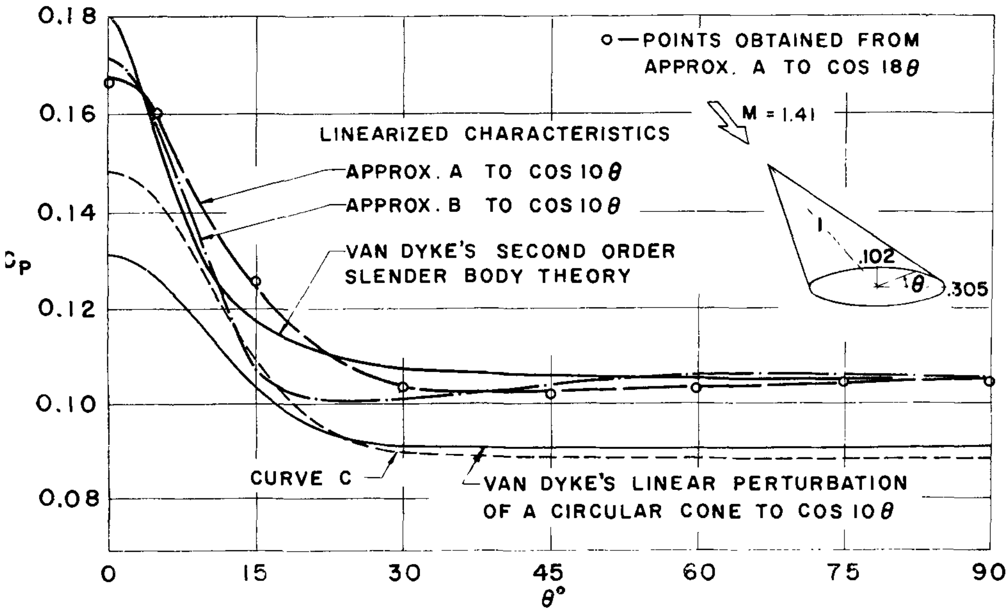

Figure 1. Comparison between various approximations to the pressure distribution around an elliptic cone.

It is shown in the present note that satisfactory results can be obtained from the method of linearized characteristics, even when only linear terms are retained in the boundary conditions, provided that an area rule requirement is satisfied. The pressure distribution predicted by this procedure for the aforementioned elliptic cone at $M = \sqrt{2}$ compares favourably with the second-order results (see figure 1); moreover, the experimental data presented in figure 3 of Van Dyke's paper indicate that the actual values in the region of small θ ($\tan\theta = b/a \tan\eta$ in the notation of his paper) should fall between the two predictions. Thus both analyses exhibit approximately the same accuracy at the surface of the body, even though the particular example (a slender body at fairly low Mach number) represents an unfavourable test for Ferri's method. It is also pertinent to observe that Van Dyke's solution is not uniformly valid throughout the flow field and fails at the shock, where conversely the accuracy of the linearized characteristics method is best since the perturbations there are smallest.

The area-rule requirement for the expansion of the boundary conditions is justified by the following considerations. If, following Ferri, the shape of the body cross-section is expressed in polar coordinates (r, ψ, θ) as the Fourier series

$$\psi_b = \psi_{0b} + \sum_{n=1}^{\infty} \psi_{nb} \cos n\theta \tag{1}$$

with only linear terms retained in ψ_{nb}, and if ψ_{0b} refers to the basic circular cone, the velocity components at the surface of the actual body are given by

$$\left.\begin{aligned} v_r &= (v_{r0})_{\psi_{0b}} \left[1 + 2 \sum_{n=1}^{\infty} \psi_{nb} \left(\frac{v_{rn}}{v_{nn}} \right)_{\psi_{0b}} \cos n\theta \right], \\ v_n &= 0, \\ w &= 2(v_{r0})_{\psi_{0b}} \sum_{n=1}^{\infty} n\psi_{nb} \left(\frac{w_n}{v_{nn}} \right)_{\psi_{0b}} \sin n\theta. \end{aligned}\right\} \tag{2}$$

In the notation used by Ward (1955) the boundary conditions are

$$\frac{v_\nu}{v_z} = \frac{d\nu}{dz} \quad \text{or} \quad \int_C \frac{v_\nu}{v_z}\, d\tau = \int_C \frac{d\nu}{dz}\, d\tau = S'(z), \tag{3}$$

where ν is the distance along the direction normal to the body contour in a plane perpendicular to the free-stream direction z, and v_ν, v_z are the velocity components in the ν-direction and z-direction respectively. Within the approximation of the linearized characteristics method, we have

$$\left.\begin{aligned} v_\nu &= (v_{r0})_{\psi_{0b}} \sin\psi_{0b} \left\{ 1 + \sum_{n=1}^{\infty} \psi_{nb} \left[2\left(\frac{v_{rn}}{v_{nn}} \right)_{\psi_{0b}} + \cot\psi_{0b} \right] \cos n\theta \right\}, \\ v_z &= (v_{r0})_{\psi_{0b}} \cos\psi_{0b} \left\{ 1 + \sum_{n=1}^{\infty} \psi_{nb} \left[2\left(\frac{v_{rn}}{v_{nn}} \right)_{\psi_{0b}} - \tan\psi_{0b} \right] \cos n\theta \right\}. \end{aligned}\right\} \tag{4}$$

Substitution of these expressions into (3) yields

$$2\pi z \tan^2\psi_{0b} = S'(z).$$

This result can alternatively be obtained by observing that in a neighbourhood $O[\psi_{nb}]$ of the body surface the following quantities are small of the same order (the symbol $O[\ ]$ indicating order of magnitude, and s entropy)

$$O[v_n] \equiv O[w] \equiv O\left[\frac{1}{\gamma R} \frac{\partial s}{\partial \psi} \right] \equiv O[\psi_{nb}].$$

If terms $O[\psi_{nb}^2]$ are neglected, the equation of motion in spherical coordinates reduces to the incompressible form

$$\operatorname{div} \mathbf{v} = 2v_r + v_n \cot\psi + \frac{\partial v_n}{\partial \psi} + \frac{\partial w}{\sin\psi\, \partial\theta} = 0,$$

and the flow in the neighbourhood in question can be defined by an incompressible potential since the entropy is constant on the body and the effect of $\partial s/\partial\psi$ on the velocities can be neglected, being $O[\psi_{nb}^2]$. Expansion of this potential in spherical harmonics and subsequent application of Gauss's theorem to a control volume delimited by the actual body, by the basic axisymmetric body and by two spherical surfaces of arbitrary radius leads to the same requirement for the zeroth term as quoted above. Therefore, the coefficients of the Fourier series representing the cross-section of the body within the approximation of the linearized characteristics method should be obtained by an expansion satisfying the requirement of equivalent areas. For the particular case of the elliptic cone at zero angle of attack, this can be obtained either (i) by determining b in the equation

$$\psi = \tan^{-1}\frac{b(a/b)}{[(a/b)^2\sin^2\theta+\cos^2\theta]^{1/2}},$$

so as to satisfy the condition $\pi\tan^2\psi_{0b} = S$ (approximation A in figure 1), or (ii) by determining ψ from the expansion

$$\tan^2\psi = \frac{a^2b^2}{a^2\sin^2\theta+b^2\cos^2\theta} = ab\left[1+2\sum_{n=1}^{\infty}\left(\frac{a-b}{a+b}\right)^n\cos 2n\theta\right],$$

which, within the linearized approximation, gives (approximation B in figure 1)

$$\tan^2\psi_{0b} = ab; \quad \psi = \psi_{0b}+\sin\psi_{0b}\cos\psi_{0b}\sum_{n=1}^{\infty}\left(\frac{a-b}{a+b}\right)^n\cos 2n\theta.$$

The expansion used in method (ii) has been suggested in a private communication by Van Dyke.

The poor accuracy exhibited by the results of Van Dyke's linear perturbation of a circular cone depended on the use of a direct Fourier expansion for the body geometry which yields $\psi_{0b} = 9{\cdot}27°$ instead of $\psi_{0b} = 10°$; in the chosen conical reference system, higher-order terms in ψ_{nb} were thus considered for the boundary conditions, while the linearized approximation was retained for the computation of the velocity components. In this connection it is interesting to observe the close agreement over the range $20° < \theta < 90°$ between Van Dyke's approximate results and the curve obtained by imposing his boundary conditions in the linearized characteristic analysis (curve C in figure 1). The increasing discrepancy between the aforesaid curves in the range $0° < \theta < 20°$ indicates that linear perturbations of the basic flow field lead to different accuracy in the two cases. This behaviour may in turn be attributed to the fact that the basic flow used in the linearized characteristics approach represents a uniformly valid solution, in contrast to the basic flow considered in Van Dyke's analysis.

The present example of a linearized characteristics calculation has also been extended to include terms up to $\cos 18\theta$ *. The results, which are

* Tables of the disturbance velocities up to $\cos 18\theta$ for a basic 10°-cone have been computed by the Research Department of the Grumman Aircraft Engineering Company.

included in figure 1, are practically identical to those obtained by the approximation as far as $\cos 10\theta$, and thus indicate that the question of convergence of the Fourier series in Ferri's method is related to the problem of representing the body cross-section rather than the problem of describing the perturbation flow field. It is believed that Van Dyke's expansion of the slender-body pressure distribution to $\cos 10\theta$ afforded a lower accuracy than that obtained in the present analysis because the disturbance velocities considered in the two cases are of different order of magnitude; indeed, the latter expands the differences in velocity components (referred to conical coordinates) between the actual and the basic non-linear conical flow, while the former involves the changes in velocity from free-stream magnitude and direction.

Part II. (By M. D. V. D.)

Professor Vaglio-Laurin makes the interesting suggestion that the range of applicability of Ferri's method may be increased by modifying it in accord with the equivalence (or 'area') rule due to Oswatitsch (1952). The modification involves the same linearization as Ferri's original method, differing from it only in higher-order terms; so that a demonstration of its superiority cannot be based on the linearized equations. Instead, one must compare with a more exact (non-linear) theory or with experiment, both of which will be done here.

First, in keeping with the spirit of my original paper (Van Dyke 1956), the second-order slender-body solution derived there has been used as the basis for a test of Vaglio-Laurin's proposal. That is, the second-order slender-body approximation is taken as a model of the full solution, and the simplifications of Ferri's method and Vaglio-Laurin's modification are introduced as additional approximations. The results are shown in figure 2 (which is the left half of figure 5 of my previous paper with two added curves). Vaglio-Laurin's approximation A is seen to give a definite improvement in the general level of pressure, though the peak is blurred (so that at this moderate Mach number simple slender-body theory is more accurate, as shown in figure 5 of my previous paper).

Second, Jorgensen (1957) has recently measured the pressures over an elliptic cone close in shape to that just considered. It has the same axis ratio of 3 : 1; and although its cross-sectional area is that of a 7·76° rather than a 10° circular cone, it was tested at a Mach number of 1·97 rather than 1·41, so that the supersonic similarity parameter (the ratio of cone angle to free-stream Mach angle) is actually somewhat higher. The results are shown in figure 3. Again the actual pressure peak is seen to be rounded off by approximation A.

Vaglio-Laurin's last paragraph is quite correct. Curtailment of the Fourier series at $\cos 10\theta$ introduces no appreciable error; and § 3.5 of my previous paper was wrong on this point for just the reason that he suggests. Hence my previous assessment of Ferri's method stands except that the

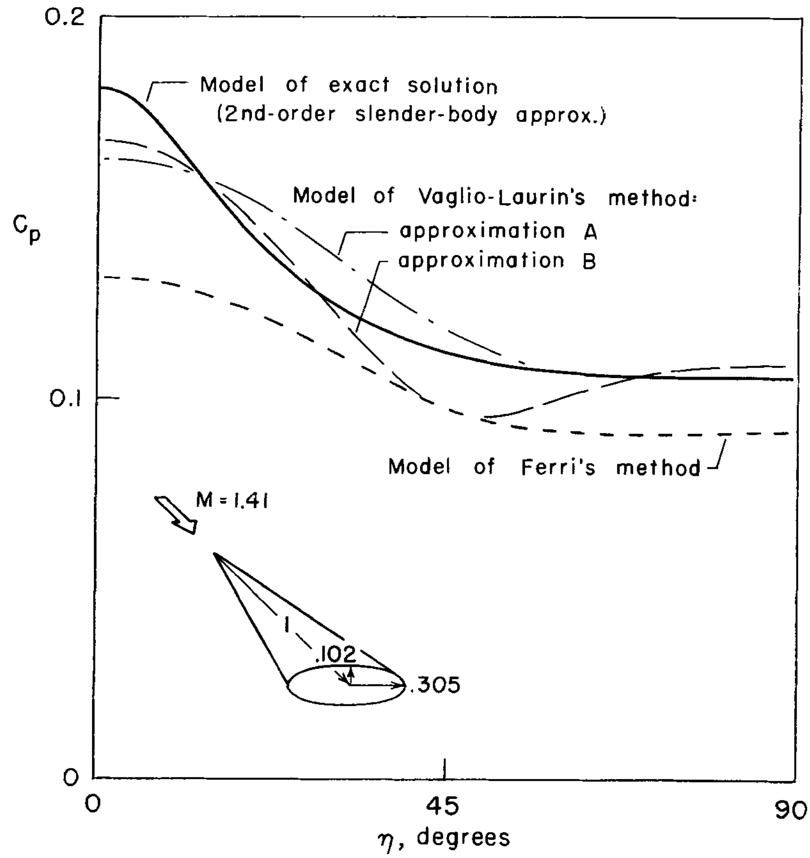

Figure 2. Further comparison between approximations to the pressure distribution around an elliptic cone.

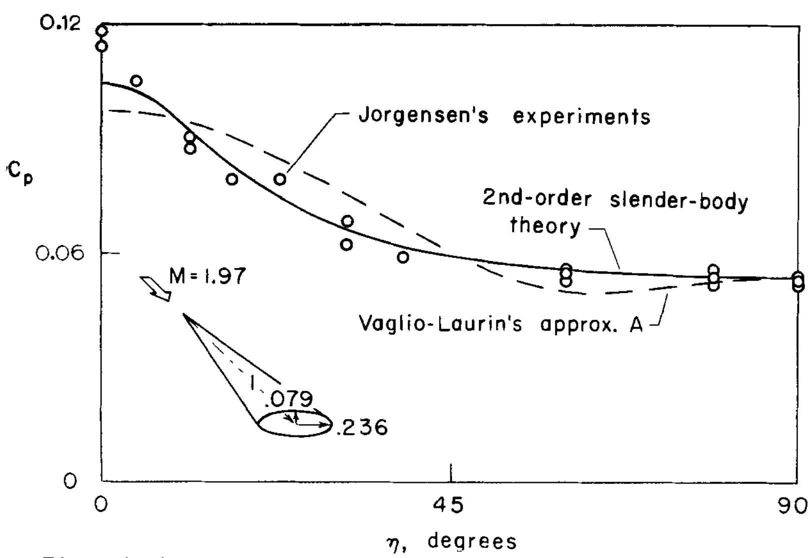

Figure 3. Comparison between theoretical and experimental results.

error is to be attributed almost entirely to non-linear effects rather than partly to the curtailment of the series. The method in its original form cannot reasonably be applied to cones whose cross-section differs from circular by as much as a 3 : 1 ellipse. For the modification suggested by Vaglio-Laurin, however, such eccentricity is seen to be perhaps tolerable at high supersonic Mach numbers where no more accurate theory exists. For less extreme eccentricities Vaglio-Laurin's method will yield good accuracy.

In view of this conclusion, the numerical results of Ferri, Ness & Kaplita (1953) must be regarded as unreliable except for nearly circular bodies. For example, the drag of an elliptic cone decreases much less with flattening than is suggested there. Indeed, Jorgensen's measurements at Mach numbers of 1·97 and 2·94 show that the decrease in wave drag is so slight as to be entirely offset by the increase in skin friction due to greater surface area.

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